JOURNAL OF APPROXIMATION THEORY 5, 15–40 (1972)

On Spline Functions Determined by Singular Self-Adjoint Differential Operators*

JOSEPH JEROME

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

AND

JOHN PIERCE

Department of Mathematics, University of Southern California, Los Angeles, California 90007

Communicated by T. J. Rivlin

Received March 20, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75th BIRTHDAY

INTRODUCTION

The subject of spline functions has developed extremely rapidly during the last ten to fifteen years and, perhaps, the major reason for this is the dual nature of the spline. On the one hand, it has been historically characterized as a real-valued function with global smoothness properties pieced together locally by null solutions of certain differential operators. On the other hand, it can frequently be characterized as the solution of a minimization problem subject to certain linear constraints. The approach taken in this paper is the variational approach and the functional $B(u, u)$ which is minimized is the bilinear form associated with an arbitrary self-adjoint operator Λ which is permitted to be singular, provided the leading coefficient has an integrable reciprocal and vanishes only at the endpoints.

This paper then unifies and extends certain concepts in Lucas [9] where nonsingular self-adjoint operators are considered, in Ciarlet, Natterer and Varga [2] where Hermite splines associated with singular self-adjoint operators are considered, and in Dailey [3] which is an extension of [2]. Moreover, it contains as special cases most of the notions of spline functions previously defined by means of nonnegative bilinear forms $B(u, u)$, associated with certain nonsingular self-adjoint ordinary differential operators, with linear

* Research supported in part by an Office of Naval Research Grant. Sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462.

equality constraints (cf. [7, 8]). In particular, it is shown in this paper that a spline function may be obtained whenever $B(u, u)$ is nonnegative over a suitable function space for the broad class of singular self-adjoint operators considered here.

The paper is divided into three sections, the first of which treats the general problem of existence. Here it is shown that a solution of the minimization problem need not exist (Theorem 2) in all cases. A sufficient condition for existence in terms of the eigenfunctions of an operator which is a natural translate of Λ is given in Theorem 1 and from this follows the existence result in the case when $B(u, u)$ is nonnegative. Theorem 3 concludes this section by showing that existence of the spline always holds if, included among the linear equality constraints, are sufficiently many-point evaluations, or derivative evaluations. Thus, Theorem 3 provides a practical existence criterion to supplement the abstract criterion of Theorem 1.

Section 2 of the paper provides an analytical characterization of the spline, illustrating the previously mentioned duality, in the case where the linear equality constraints are so-called extended Hermite–Birkhoff functionals. The principal result is described in Theorem 4 which extends corresponding concepts in [7].

Section 3 of the paper deals with error estimates, or, more precisely, the order of approximation of the splines to smooth functions whose higher order derivatives are square integrable with respect to a weight determined by the leading coefficient of Λ. These results reduce, in the nonsingular case, to those of [8] and comprise the content of Theorems 5–8. These convergence results are especially useful in the use of Galerkin subspaces to approximate solutions of two-point boundary value problems.

I. Existence

Let Λ be the formally self-adjoint $2n$-th order differential operator

$$\Lambda \equiv \sum_{j=0}^{n} (-1)^j D^j[a_j(x) D^j], \qquad D \equiv \frac{d}{dx}, \tag{1.1}$$

where we assume that

$$\begin{aligned} &\text{(i) } a_n(x) > 0 \text{ for all } x \in (a, b), \\ &\text{(ii) } 1/a_n \in L^1[a, b], \\ &\text{(iii) } a_j \in C^j[a, b], \qquad 0 \leqslant j < n, \qquad a_n \in C^n(a, b). \end{aligned} \tag{1.2}$$

Let H denote the linear space of all real-valued functions f, defined on

$[a, b]$, such that $D^{n-1}f$ is absolutely continuous, and $\sqrt{a_n}\, D^n f \in L^2[a, b]$. Define the bilinear form $B(u, v)$ as follows:

$$B(u, v) \equiv \sum_{j=0}^{n} \int_a^b a_j(x)\, D^j u(x)\, D^j v(x)\, dx, \qquad \text{for all} \quad u, v \in H. \tag{1.3}$$

It follows easily that $B(u, v)$ is defined for all $u, v \in H$, and moreover, that there exists a constant C', possibly negative, such that

$$B(u, u) - \int_a^b a_n(x)[D^n u(x)]^2\, dx \geqslant C' \| u \|_{n-1,2}^2 \qquad \text{for all} \quad u \in H, \tag{1.4}$$

where $\| \cdot \|_{n-1,2}$ is the usual Sobolev $n-1, 2$ norm, defined from

$$(u, v)_{j,2} \equiv \int_a^b \left(\sum_{k=0}^{j} D^k u(x)\, D^k v(x) \right) dx. \tag{1.5}$$

We now show that there exist positive constants α and C such that

$$B(u, u) + C \| u \|_{n-1,2}^2 \geqslant \alpha \left[\int_a^b a_n(x)[D^n u(x)]^2\, dx + \sum_{j=0}^{n-1} [D^j u(a)]^2 \right]$$

$$\text{for all } u \in H. \tag{1.6}$$

From the identity

$$D^{n-1}u(a) = \left(\frac{1}{b-a} \right) \left\{ - \int_a^b \left(\int_a^x D^n u(t)\, dt \right) dx + \int_a^b D^{n-1}u(x)\, dx \right\},$$

it follows from the Schwarz inequality and (1.2ii) that

$$\begin{aligned} | D^{n-1}u(a)|^2 & \\ &\leqslant \frac{2}{(b-a)} \left(\| D^{n-1}u \|_{0,2}^2 + (b-a) \left[\int_a^b \frac{1}{a_n(x)}\, dx \right] \int_a^b a_n(x)[D^n u(x)]^2\, dx \right) \\ &\leqslant \frac{2}{(b-a)} \left(\| u \|_{n-1,2}^2 + K \int_a^b a_n(x)[D^n u(x)]^2\, dx \right). \end{aligned}$$

From Sobolev's inequality [1, p. 32], there exists a positive constant γ such that

$$\gamma \| u \|_{n-1,2}^2 \geqslant \sum_{j=0}^{n-2} [D^j u(a)]^2.$$

The result of (1.6) then follows from (1.4) and the above two inequalities.

We will now show that the following inequality is satisfied.

$$B(u, u) + C \| u \|_{j_0,2}^2 \geqslant \alpha \left[\int_a^b a_n(x)[D^n u(x)]^2 \, dx + \sum_{j=0}^{n-1} [D^j u(a)]^2 \right], \tag{1.7}$$

for all $u \in H$, for all $0 \leqslant j_0 \leqslant n - 1$, where α and $C = C_{j_0}$ are positive constants. Indeed, (1.7) for $j_0 = 0$ (which implies (1.7) for $j_0 \geqslant 0$) follows from (1.6) and [12, Theorem 1] the inequality, valid for $0 < \epsilon \leqslant 1$, and $0 \leqslant j < n$:

$$\| u \|_{j,2}^2 \leqslant \gamma(\epsilon^{n-j-\frac{1}{2}} \| \sqrt{a_n}\, D^n u \|_{0,2}^2 + \epsilon^{-j} \| u \|_{0,2}^2), \ u \in H[a, b],[1]$$

for some positive constant γ. Since the constant C of (1.7) may be expected to decrease with increasing j_0, we retain the flexibility of (possibly) choosing $j_0 > 0$ in what follows.

We now define an inner product on the space H as follows.

$$(u, v)_D \equiv B(u, v) + C(u, v)_{j_0,2}, \tag{1.8}$$

with j_0 and C as in (1.7). To show that H is a Hilbert space under $(\cdot, \cdot)_D$, we first show that there exist positive constants K_j, $0 \leqslant j \leqslant n - 1$, such that

$$\| u \|_D^2 \geqslant K_j \| D^j u \|_{L^\infty}^2, \qquad 0 \leqslant j \leqslant n - 1, \qquad \text{for all} \quad u \in H. \tag{1.9}$$

From (1.7) it is clear that

$$\| u \|_D^2 \geqslant \alpha \int_a^b a_n(x)[D^n u(x)]^2 \, dx + \alpha \sum_{j=0}^{n-1} [D^j u(a)]^2 \qquad \text{for all} \quad u \in H. \tag{1.10}$$

Since $D^{n-1}f$ is absolutely continuous,

$$D^{n-1}f(t) = \int_a^t D^n f(x) \, dx + D^{n-1}f(a), \text{ and therefore}$$

$$| D^{n-1}f(t)|^2 \leqslant 2 \left[\left[\int_a^t D^n f(x) \, dx \right]^2 + [D^{n-1}f(a)]^2 \right].$$

However,

$$\begin{aligned} \left[\int_a^t D^n f(x) \, dx \right]^2 &\leqslant \left[\int_a^b \left| \frac{1}{\sqrt{a_n(x)}} \cdot \sqrt{a_n(x)}\, D^n f(x) \right| dx \right]^2 \\ &\leqslant \left\| \frac{1}{a_n} \right\|_{L^1} \cdot \int_a^b a_n(x)[D^n f(x)]^2 \, dx \\ &\equiv K \int_a^t a_n(x)[D^n f(x)]^2 \, dx, \end{aligned}$$

[1] A version of this inequality has been obtained independently by M. Silverstein (unpublished).

where we have used Cauchy–Schwarz and (1.2i)–(1.2ii). Thus, from (1.10)

$$| D^{n-1}f(t)|^2$$
$$\leqslant 2 \left[K \int_a^b a_n(x)[D^n f(x)]^2\, dx + \sum_{j=0}^{n-1} [D^j f(a)]^2\right] \leqslant \frac{2}{\alpha} \max(K, 1) \| u \|_D^2 .$$

Since this holds for all $t \in [a, b]$, we have proven (1.9) for $j = n - 1$. The inequalities for $0 \leqslant j \leqslant n - 2$ follow similarly. From (1.9) and (1.10) it is clear that H is a Hilbert space under $\| \cdot \|_D$, since convergence in $\| \cdot \|_D$ implies L^∞ convergence in all derivatives to order $n - 1$, and convergence of the n-th derivative in the weighted L^2-norm. Finally, from (1.9), it then follows that there exist positive constants K_j', $0 \leqslant j \leqslant n - 1$, such that

$$\| u \|_D^2 \geqslant K_j' \| u \|_{j,2}^2 \qquad \text{for all} \quad u \in H, \qquad 0 \leqslant j \leqslant n - 1. \tag{1.11}$$

Let $M = \{\mu_j\}_{j=1}^m$ be any set of bounded linear functionals, linearly independent and continuous over H, and $\bar{r} \in E^m$ an m-vector of real numbers. We then make the following definition.

DEFINITION 1. $s \in H$ is called a Λ-spline, interpolating $\bar{r}$ with respect to $M = \{\mu_j\}_{j=1}^m$, if s solves the following minimization problem:

$$B(s, s) = \min_{u \in U(r)} B(u, u), \tag{1.12}$$

where

$$U(\bar{r}) \equiv \{u \in H : \mu_j u = r_j\,,\ 1 \leqslant j \leqslant m\}.$$

We remark that, as a consequence of (1.9), point evaluations of a function and its derivatives up to order $n - 1$ are continuous linear functionals over the space H. We shall be initially concerned with proving the existence of Λ-splines. In order to do so in the most general manner, we shall require some technical results.

By definition, the space H is contained in $W^{j,2}[a, b]$, for each $0 \leqslant j \leqslant n - 1$. We now show that the injection of H into $W^{j,2}[a, b]$ is compact for each $0 \leqslant j \leqslant n - 1$. Suppose $\{u_m\}_{m=1}^\infty$ is a sequence in H such that $\| u_m \|_D \leqslant c$ for all $m \geqslant 1$. We must show there is a subsequence $\{u_{m_k}\}_{k=1}^\infty$ convergent in $W^{j,2}[a, b]$. From (1.9), there exists a constant c' such that $\{\| D^j u_m \|_{L^\infty}\} \leqslant c'$, for all $m \geqslant 1$, $0 \leqslant j \leqslant n - 1$. Thus, the sequence $\{D^{n-1} u_m\}_{m=1}^\infty$ is uniformly bounded.

Moreover,

$$
\begin{aligned}
| D^{n-1}u_m(y) &- D^{n-1}u_m(x)|^2 \\
&= \left|\int_x^y D^n u_m(t)\,dt\right|^2 \\
&\leqslant \left[\int_x^y \left|\frac{1}{\sqrt{a_n(t)}}\cdot\sqrt{a_n(t)}\,D^n u_m(t)\right|dt\right]^2 \\
&\leqslant \int_x^y \frac{1}{a_n(t)}\,dt\cdot\int_x^y a_n(t)[D^n u_m(t)]^2\,dt\Big]
\end{aligned}
$$

which, from (1.10), becomes

$$| D^{n-1}u_m(y) - D^{n-1}u_m(x)|^2 \leqslant \frac{c^2}{\alpha}\int_x^y \frac{1}{a_n(t)}\,dt.$$

Since $1/a_n \in L^1[a, b]$, it follows that $\int_x^y [1/a_n(t)]\,dt$ is less than ϵ whenever $| x - y | < \delta(\epsilon, a_n)$, and hence the family $\{D^{n-1}u_m\}_{m=1}^{\infty}$ is also equicontinuous. Therefore, there exists a subsequence $\{D^{n-1}u_{m_{k_1}}\}_{k_1=1}^{\infty}$ which is convergent in L^∞, and hence in L^2. Now consider the subsequence $\{D^{n-2}u_{m_{k_1}}\}_{k_1=1}^{\infty}$. Again this is a uniformly bounded, equicontinuous sequence, and we can extract a subsequence convergent in L^2. Continuing this process, we obtain a subsequence of the $\{u_m\}_{m=1}^{\infty}$ which is convergent in $W^{j,2}[a, b]$ for all $0 \leqslant j \leqslant n - 1$. Thus, in particular, for j_0 as in (1.7)

$$I : H \to W^{j_0,2} \qquad \text{is compact,} \tag{1.13}$$

where I is the injection mapping of H into $W^{j_0,2}$.

With j_0 as in (1.7), it follows from (1.11) that

$$(u, u)_D \geqslant K \| u \|_{j_0,2}^2 \qquad \text{for all} \quad u \in H, \tag{1.14}$$

for some constant $K > 0$. Thus, from the Friedrichs extension theorem [10, p. 335], there exists a self-adjoint positive definite transformation

$$A : \mathscr{D}_A \subset H \xrightarrow{\text{onto}} W^{j_0,2} \text{ such that}$$

$$(u, v)_D = (Au, v)_{j_0,2} \qquad \text{for all} \quad u \in \mathscr{D}_A,\ v \in H. \tag{1.15}$$

The domain $\mathscr{D}_A$ of A is a linear subspace, dense in H. Moreover, A^{-1} exists and is a self-adjoint operator from $W^{j_0,2} \to \mathscr{D}_A$, and, from (1.13), is *compact* when viewed as a transformation from $W^{j_0,2} \to W^{j_0,2}$. Thus A has a discrete spectrum $\{\lambda_j\}_{j=1}^{\infty}$ such that $\lambda_j \to \infty$ as $j \to \infty$, and the eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$ of A are orthonormal and complete in $W^{j_0,2}$. Moreover, since

$$(\varphi_i, \varphi_j)_D = (A\varphi_i, \varphi_j)_{j_0,2} = \lambda_i(\varphi_i, \varphi_j)_{j_0,2} = \lambda_i \delta_{ij},$$

the $\{\varphi_j\}_{j=1}^{\infty}$ are orthogonal in H with respect to $(\cdot, \cdot)_D$. The set $\{\varphi_j\}_{j=1}^{\infty}$ is also complete in H, since if $(\varphi_i, w)_D = 0$ for all i, then $(\varphi_i, w)_D = (A\varphi_i, w)_{j_0,2} = \lambda_i(\varphi_i, w)_{j_0,2} = 0$. Since $\lambda_i \neq 0$ for all i, then $w = 0$.

Since A may not be defined on all of H, we want to show that

$$(u, v)_D = (A^{1/2}u, A^{1/2}v)_{j_0,2}, \qquad \text{for all} \quad u, v \in H. \tag{1.16}$$

Consider the transformation $A^{1/2}: \mathscr{D}_{A^{1/2}} \to W^{j_0,2}$. Since A is self-adjoint, so is $A^{1/2}$ on $\mathscr{D}_{A^{1/2}} = \{u \in W^{j_0,2}: \sum_j \lambda_j(u, \varphi_j)_{j_0,2}^2 < \infty\} \supset \mathscr{D}_A$. But since

$$u = \sum_j \frac{(u, \varphi_j)_D}{\lambda_j} \varphi_j \qquad \text{for all} \quad u \in H, \qquad \text{with} \quad \sum_j \frac{(u, \varphi_j)_D^2}{\lambda_j} < \infty,$$

it follows from the relation $(u, \varphi_j)_D/\lambda_j = (u, \varphi_j)_{j_0,2}$ that $A^{1/2}$ has domain containing H. It then follows that

$$(u, v)_D = (A^{1/2}u, A^{1/2}v)_{j_0,2} \qquad \text{for all} \quad u, v \in H.$$

Now define the operator $\mathscr{M}$ by

$$\mathscr{M}u = \sum_{i=0}^{0} (-1)^i D^{2i}u \qquad \text{for} \quad u \in H \cap C^{2j_0}(a, b) \tag{1.17}$$

Then from the relation

$$(\varphi, u)_D = B(\varphi, u) + C(\varphi, u)_{j_0,2}$$

we have, for each $u \in C_c^{\infty}(a, b)$ and each eigenfunction φ of A corresponding to eigenvalue λ,

$$B(\varphi, u) + C(\varphi, u)_{j_0} - \lambda(\varphi, u)_{j_0} = 0,$$

i.e.,

$$(\varphi, \Lambda u + C\mathscr{M}u - \lambda\mathscr{M}u)_{0,2} = 0$$

which shows [6, Sec. 8] that λ is an eigenvalue and φ a $C^{2n}(a, b)$ eigenfunction of

$$(\Lambda + C\mathscr{M})\varphi = \lambda\mathscr{M}\varphi, \qquad \varphi \in \mathscr{D}_A \cap C^{2n}(a, b). \tag{1.18}$$

(1.18) thus has a discrete positive spectrum and a complete orthogonal system of eigenfunctions in H and $W^{j_0,2}$, which coincide with the spectrum and eigenfunctions of A.

We are now prepared to state a basic theorem concerning the existence of Λ-splines.

THEOREM 1. *Let $M = \{\mu_j\}_{j=1}^m$ be any set of linear functionals, linearly independent and continuous over H. If there exist eigenvalues $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_{r-1} < C$ and $\lambda_r = \cdots = \lambda_{t-1} = C$ of* (1.18), *with C as in* (1.7), *let $\varphi_1, \ldots, \varphi_{t-1}$ be the corresponding eigenfunctions. Then the minimization problem* (1.12) *has a solution for all $\bar{r} \in E^m$ if $U(\bar{0})$ is orthogonal in H to $\{\varphi_1, \ldots, \varphi_{r-1}\}$. In this case the solution is unique if and only if $U(\bar{0}) \cap$ span $\langle \varphi_r, \ldots, \varphi_{t-1} \rangle = \{0\}$. In particular, if $B(u, u) \geqslant 0$ for all $u \in H$, then a solution of* (1.12) *always exists. If $j_0 = 0$ then $\mathscr{M} = I$ in* (1.18).

Proof. The subspace $U(\bar{0})$ is closed in H and of finite co-dimension. For a given $\bar{r} \in E^m$, choose any element f_0 of $U(\bar{r})$ (such an f_0 exists because of the linear independence of the $\{\mu_j\}_{j=1}^m$) and hold it fixed. The hyperplane $\{f_0 - u \colon u \in U(\bar{0})\}$ is also closed, and of finite co-dimension. Solving (1.12) is equivalent to solving the following problem

$$B(s, s) = \min_{u \in U(\bar{0})} B(f_0 - u, f_0 - u). \tag{1.19}$$

Write the orthogonal decomposition of f_0 and $u \in U(\bar{0})$:

$$\begin{aligned} f_0 &= f_1 + f_2 + f_3 \qquad && \text{where } f_1, u_1 \in \text{span} \langle \varphi_1, \ldots, \varphi_{r-1} \rangle \\ u &= u_1 + u_2 + u_3 && \phantom{\text{where }} f_2, u_2 \in \text{span} \langle \varphi_r, \ldots, \varphi_{t-1} \rangle \\ & && \phantom{\text{where }} f_3, u_3 \in \text{span} \langle \varphi_t, \ldots \rangle. \end{aligned}$$

Then $u_1 \equiv 0$ by our assumption on M, and therefore

$$\begin{aligned} B(f_0 - u, f_0 - u) &= \| f_0 - u \|_D^2 - C \| f_0 - u \|_{j_0,2}^2 \\ &= \| f_1 \|^2 - C \| f_1 \|_{j_0,2}^2 \\ &\quad + \| f_2 - u_2 \|_D^2 - C \| f_2 - u_2 \|_{j_0,2}^2 \\ &\quad + \| f_3 - u_3 \|_D^2 - C \| f_3 - u_3 \|_{j_0,2}^2 . \end{aligned}$$

But the terms on the first line of the decomposition are constant, and the term on the second line is zero. Thus, to show (1.19) has a solution, we need merely show that there exists a minimum of the quantity

$$\| f_3 - u_3 \|_D^2 - C \| f_3 - u_3 \|_{j_0,2}^2$$

over the set $U_3(\bar{0}) \equiv \{u_3$: there exists $u_2 = \sum_{i=r}^{t-1} \alpha_i \varphi_i$ such that $u_2 + u_3 \in U(\bar{0})\}$. Let

$$H_t \equiv \left\{ f : f = g - \sum_{i=1}^{t-1} \frac{(g, \varphi_i)_D}{\lambda_i} \varphi_i , \quad g \in H \right\}$$

be the orthogonal complement of $\langle \varphi_1 ,..., \varphi_{t-1} \rangle$ in H. Let P_t be the projection of H onto H_t, i.e.,

$$P_t f = f - \sum_{i=1}^{t-1} \frac{(f, \varphi_i)_D}{\lambda_i} \varphi_i \qquad \text{for all} \quad f \in H.$$

Then $P_t U(\bar{0}) = U_3(\bar{0})$ and since the null space of P_t is a finite dimensional space it follows [4, Lemma 2.1] that $U_3(\bar{0})$ is closed and convex. Moreover H_t is also a Hilbert space under the norm $\| \cdot \|_t$ defined from $\| \cdot \|_t^2 \equiv \| \cdot \|_D^2 - C \| \cdot \|_{j_0,2}^2$. For, $\lambda_t > C$ and

$$\| f \|_D^2 = (A^{1/2} f, A^{1/2} f)_{j_0,2} \geqslant \lambda_t \| f \|_{j_0,2}^2 \qquad \text{for all} \quad f \in H_t .$$

Therefore,

$$\begin{aligned} \| f \|_D^2 - C \| f \|_{j_0,2}^2 &= \left(1 - \frac{C}{\lambda_t}\right) \| f \|_D^2 + \frac{C}{\lambda_t} \| f \|_D^2 - C \| f \|_{j_0,2}^2 \\ &\geqslant \left(1 - \frac{C}{\lambda_t}\right) \| f \|_D^2 = C_1 \| f \|_D^2 \qquad \text{for all} \quad f \in H_t , \end{aligned}$$

where C_1 is a positive constant. It is immediate, moreover, that

$$\| f \|_D^2 \geqslant \| f \|_D^2 - C \| f \|_{j_0,2}^2 \qquad \text{for all} \quad f \in H_t .$$

Thus the norms $\| \cdot \|_D$ and $\| \cdot \|_t$ are equivalent on H_t. Therefore, the set $f_3 - U_3(\bar{0})$ is a closed and convex subset of H_t under the norm $\| \cdot \|_t$, and the quantity

$$\| f_3 - u_3 \|_t^2 \equiv \| f_3 - u_3 \|_D^2 - C \| f_3 - u_3 \|_{j_0,2}^2$$

possesses a unique minimum over $U_3(\bar{0})$, as was to be shown. Thus (1.12) has a solution s for any $\bar{r} \in E^m$. If s' is any solution of (1.12), it follows from the above arguments that $s - s' \in \operatorname{span} \langle \varphi_r ,..., \varphi_{t-1} \rangle$. Moreover, $\mu(s - s') = 0$ for all $\mu \in M$, and hence $s - s' \in U(\bar{0}) \cap \operatorname{span} \langle \varphi_r ,..., \varphi_{t-1} \rangle$. The spline s is therefore unique if $U(\bar{0}) \cap \operatorname{span} \langle \varphi_r ,..., \varphi_{t-1} \rangle = \{0\}$. Conversely, if $u \in U(\bar{0}) \cap \operatorname{span} \langle \varphi_r ,..., \varphi_{t-1} \rangle$, then $s + u$ also satisfies (1.12). Hence, if s is unique, $u = 0$ and $U(\bar{0}) \cap \operatorname{span} \{\varphi_r ,..., \varphi_{t-1}\} = \{0\}$.

If $B(u, u) \geqslant 0$ then clearly

$$B(u, u) + C \| u \|_{j_0,2}^2 \geqslant C \| u \|_{j_0,2}^2 .$$

Hence all the eigenvalues of the operator A are greater than or equal to C and a solution of (1.12) always exists.

Finally, the last statement of the theorem follows from earlier remarks.

COROLLARY 1. *If $s \in U(\bar{r})$ is a solution of* (1.12), *then*

$$B(s, h) = 0 \qquad \textit{for all } h \in U(\bar{0}). \tag{1.20}$$

Conversely, if (1.20) *holds and $U(\bar{0})^{\perp} \supset \{\varphi_1, \ldots, \varphi_{r-1}\}$, then s solves* (1.12).

Proof. If $B(s, h) = 0$ for all $h \in U(\bar{0})$, let s_1 be any other element of $U(\bar{r})$. Then, using only the bi-linearity and symmetry of $B(u, v)$,

$$\begin{aligned} B(s_1, s_1) &= B(s, s_1) + B(s_1 - s, s_1) \\ &= B(s, s) + 2B(s, s_1 - s) + B(s_1 - s, s_1 - s). \end{aligned}$$

But, since $s_1 - s \in U(\bar{0})$, then $B(s, s_1 - s) = 0$, and by the assumption that $U(\bar{0})^{\perp} \supset \{\varphi_1, \ldots, \varphi_{r-1}\}$, $B(s_1 - s, s_1 - s) \geqslant 0$. Hence $B(s_1, s_1) \geqslant B(s, s)$ and s solves (1.12).

Let s be any Λ-spline interpolating $\bar{r}$ with respect to M. Let $h \in U(\bar{0})$. Then, since $s + \epsilon h \in U(\bar{r})$ for any ϵ,

$$B(s + \epsilon h, s + \epsilon h) = B(s, s) + 2\epsilon B(s, h) + \epsilon^2 B(h, h) \geqslant B(s, s) \text{ for any } \epsilon.$$

Hence

$$2\epsilon B(s, h) + \epsilon^2 B(h, h) \geqslant 0. \tag{1.21}$$

There are three cases, depending upon the sign of $B(h, h)$.

(a) If $B(h, h) = 0$, then (1.21) $\Rightarrow B(s, h) = 0$;

(b) If $B(h, h) < 0$, choose

$$\epsilon = \frac{B(s, h)}{B(h, h)}; \qquad \text{then} \quad (1.21) \Rightarrow \frac{3B(s, h)^2}{B(h, h)} \geqslant 0$$

and hence $B(s, h) = 0$;

(c) If $B(h, h) > 0$, choose

$$\epsilon = -\frac{B(s, h)}{B(h, h)}; \qquad \text{then} \quad (1.21) \Rightarrow -\frac{[B(s, h)]^2}{B(h, h)} \geqslant 0$$

and again $B(s, h) = 0$.

As a partial converse of Theorem 1, we prove

THEOREM 2. *If* $U(\bar{0}) \cap \operatorname{span}\langle \varphi_1, ..., \varphi_{r-1}\rangle \not\equiv \{0\}$, *where* $\{\varphi_1, ..., \varphi_{r-1}\}$ *are the eigenfunctions described in Theorem* 1, *then no solution of* (1.12) *exists for any* $\bar{r} \in E^m$.

Proof. Suppose $U(\bar{0}) \cap \operatorname{span}\langle \varphi_1, ..., \varphi_{r-1}\rangle \supset \{\varphi\}$ and suppose $u \in U(\bar{r})$ solves (1.12). Then for any constant k, $u + k\varphi \in U(\bar{r})$, and

$$B(u + k\varphi, u + k\varphi) = B(u, u) + 2kB(u, \varphi) + k^2 B(\varphi, \varphi).$$

But $B(\varphi, \varphi) = \| \varphi \|_D^2 - C \| \varphi \|_{j_0,2}^2 < 0$. There are three cases to examine.

(a) If $B(u, \varphi) = 0$ then $B(u + k\varphi, u + k\varphi) < B(u, u)$ for any $k \neq 0$;

(b) If $B(u, \varphi) < 0$ take $k > 0$; then $B(u + k\varphi, u + k\varphi) < B(u, u)$;

(c) If $B(u, \varphi) > 0$ take $k < 0$; then $B(u + k\varphi, u + k\varphi) < B(u, u)$.

Thus we arrive at a contradiction.

DEFINITION 2. Given a differential operator Λ satisfying (1.2), and a set M of linear functionals, linearly independent and continuous over H, we say that M generates a *Λ-poised* interpolation problem if there exists a solution of (1.12) for any $\bar{r} \in E^m$. $Sp(\Lambda, M)$ is defined to be the set of all functions $s \in H$ such that s satisfies (1.12) for some $\bar{r} \in E^m$.

The following result is then an easy consequence of Theorem 1.

COROLLARY 2. *Let* $\{\varphi_i\}_{i=1}^{t-1}$ *be as in Theorem* 1. *M generates a Λ-poised problem if* $U(\bar{0})^\perp \supset \{\varphi_1, ..., \varphi_{r-1}\}$. *$Sp(\Lambda, M)$ is then a finite-dimensional subspace of H of dimension* $m + \dim\{U(\bar{0}) \cap \operatorname{span}\langle \varphi_r, ..., \varphi_{t-1}\rangle\}$.

The result of Theorem 1 does not give very practical conditions for existence and uniqueness of Λ-splines in some cases. We now show that existence and uniqueness is guaranteed provided M contains "enough" point evaluations.

We first prove the following easy estimates. Let $a = x_0 \leqslant x_1 < x_2 < \cdots < x_N \leqslant x_{N+1} = b$ be a partition of the interval $[a, b]$, where $h_i \equiv x_{i+1} - x_i$, $0 \leqslant i \leqslant N$, and $\bar{\Delta} = \max_i h_i$. Then, for all $f \in H$,

$$\| D^j f \|_{L^2}^2 \leqslant 4 \sum_{i=1}^{N} [D^j f(x_i)]^2 \bar{\Delta} + 2\bar{\Delta}(b - a) \| D^{j+1} f \|_{L^2}^2, \qquad 0 \leqslant j \leqslant n - 2, \tag{1.22}$$

and

$$\| D^{n-1}f \|_{L^2}^2$$
$$\leqslant 4 \sum_{i=1}^{N} [D^{n-1}f(x_i)]^2 \bar{\Delta} + 2(b-a)\, w_1\left(\frac{1}{a_n}, \bar{\Delta}\right) \int_a^b a_n(x)[D^n f(x)]^2\, dx, \tag{1.23}$$

where

$$w_1(g, \delta) \equiv \sup_{\substack{|t| \leqslant \delta \\ x, x+t \in [a,b]}} \left| \int_x^{x+t} |g(y)|\, dy \right|.$$

For,

$$D^j f(x) = \int_{x_i}^{x} D^{j+1} f(t)\, dt + D^j f(x_i), \qquad \text{for} \quad x \in [x_i, x_{i+1}],\ 1 \leqslant i \leqslant N,$$

and

$$D^j f(x) = -\int_x^{x_1} D^{j+1} f(t)\, dt + D^j f(x_1), \qquad \text{for} \quad x \in [a, x_1].$$

Thus,

$$| D^j f(x)|^2 \leqslant 2 \left[\left[\int_{x_i}^{x} D^{j+1} f(t)\, dt \right]^2 + [D^j f(x_i)]^2 \right], \qquad 1 \leqslant i \leqslant N,$$

and

$$\| D^j f \|_{L^2[x_1,b]}^2 \leqslant \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} 2 \left[\left[\int_{x_i}^{x} D^{j+1} f(t)\, dt \right]^2 + [D^j f(x_i)]^2 \right] dx. \tag{1.24}$$

Similarly,

$$\| D^j f \|_{L^2[a,x_1]}^2 \leqslant \int_a^{x_1} 2 \left[\left[\int_{x_1}^{x} D^{j+1} f(t)\, dt \right]^2 + [D^j f(x_1)]^2 \right] dx. \tag{1.25}$$

Hence, for $0 \leqslant j \leqslant n-2$,

$$\| D^j f \|_{L^2[x_1,b]}^2 \leqslant 2(b - x_1)\, \bar{\Delta}\, \| D^{j+1} f \|_{L^2[a,b]}^2 + 2 \sum_{i=1}^{N} [D^j f(x_i)]^2\, \bar{\Delta},$$

and

$$\| D^j f \|_{L^2[a,x_1]}^2 \leqslant 2(x_1 - a)\, \bar{\Delta}\, \| D^{j+1} f \|_{L^2[a,b]}^2 + 2[D^j f(x_1)]^2\, \bar{\Delta},$$

from which (1.22) follows. For the case $j = n - 1$, we have that

$$\left[\int_{x_1}^{x} D^n f(t)\,dt\right]^2 = \left[\int_{x_1}^{x} \frac{1}{\sqrt{a_n(t)}} \sqrt{a_n(t)}\,[D^n f(t)]\,dt\right]^2 \leqslant w_1\left(\frac{1}{a_n}, \bar{\Delta}\right) \int_a^b a_n(t)[D^n f(t)]^2\,dt, \tag{1.26}$$

for all $x \in [x_i, x_{i+1}]$, $0 \leqslant i \leqslant N$. The result of (1.23) follows as above from (1.24)–(1.26).

THEOREM 3. *Fix a choice of j for some $0 \leqslant j \leqslant n - 1$. Let*

$$\Delta^{(j)}: a = x_0^{(j)} \leqslant x_1^{(j)} < x_2^{(j)} < \cdots < x_{N_j}^{(j)} \leqslant x_{N_j+1}^{(j)} = b$$

denote a partition of $[a, b]$ such that $\{\mu_i : \mu_i f \equiv D^j f(x_i^{(j)}),\ 1 \leqslant i \leqslant N_j\} \subset M$. Define $h_i^{(j)} \equiv x_{i+1}^{(j)} - x_i^{(j)}$, $0 \leqslant i \leqslant N_j$, and $\bar{\Delta}^{(j)} \equiv \max_i h_i^{(j)}$. Assume that there exists an $x' \in [a, b]$ such that $M \supset \{\mu_i : \mu_i f \equiv D^i f(x'),\ 0 \leqslant i \leqslant j - 1\}$. Then for $\bar{\Delta}^{(j)}$ sufficiently small, there exists a unique Λ-spline interpolating $\vec{r}$ with respect to M for each $\vec{r} \in E^m$.

Proof. We first show that for $\bar{\Delta}^{(j)}$ sufficiently small, there exists a constant $L_j > 0$ such that $\|\cdot\|_M$, defined from

$$\| u \|_M^2 \equiv B(u, u) + L_j \sum_{\mu \in M} [\mu(u)]^2 \tag{1.27}$$

is a norm on H, equivalent to $\|\cdot\|_D$.

First, there exist constants $C_j > 0$ such that

$$\| u \|_{j,2}^2 \leqslant C_j \left[\sum_{k=0}^{j-1} [D^k u(x')]^2 + \| D^j u \|_{L^2}^2\right], \qquad 0 \leqslant j \leqslant n - 1. \tag{1.28}$$

Since $\bar{\Delta}^{(j)} \leqslant b - a$, we have from (1.22) and (1.23) that

$$\| u \|_{j,2}^2 \leqslant C_j \left[\sum_{k=0}^{j-1} [D^k u(x')]^2 + 4(b - a) \sum_{i=1}^{N_j} [D^j u(x_i^{(j)})]^2 + 2\bar{\Delta}^{(j)}(b - a)\| D^{j+1} u \|_{L^2}^2\right], \qquad 0 \leqslant j \leqslant n - 2, \tag{1.29}$$

and

$$\| u \|_{n-1,2}^2 \leqslant C_{n-1} \left[\sum_{k=0}^{n-2} [D^k u(x')]^2 + 4(b - a) \sum_{i=1}^{N_{n-1}} [D^{n-1} u(x_i^{(n-1)})]^2 + 2w_1\left(\frac{1}{a_n}, \bar{\Delta}^{(n-1)}\right)(b - a) \times \int_a^b a_n(t)[D^n u(t)]^2\,dt\right]. \tag{1.30}$$

We complete the proof only in the cases $0 \leqslant j \leqslant n-2$. From (1.8), (1.29), and (1.11), it follows that

$$\|u\|_D^2 \leqslant B(u,u) + CC_j\Big[\sum_{k=0}^{j-1}[D^k u(x')]^2 + 4(b-a)\sum_{i=1}^{N_j}[D^j u(x_i^{(j)})]^2$$
$$+ 2\frac{1}{K'_{j+1}}\bar{\Delta}^{(j)}(b-a)\|u\|_D^2\Big], \qquad 0 \leqslant j \leqslant n-2.$$

Thus, there exist constants $L_j > 0$ such that

$$\|u\|_D^2 \leqslant B(u,u) + L_j \sum_{\mu\in M}[\mu(u)]^2 + L_j\bar{\Delta}^{(j)}\|u\|_D^2, \qquad 0 \leqslant j \leqslant n-2.$$

For all partitions $\Delta^{(j)}$ such that $L_j\bar{\Delta}^{(j)} \leqslant K < 1$, it follows that

$$\|u\|_M^2 \geqslant (1-K)\|u\|_D^2 .$$

By the continuity of the functionals of M, and the definition of $\|\cdot\|_D$, it then follows immediately that the norms $\|\cdot\|_M$ and $\|\cdot\|_D$ are equivalent.

This result follows similarly if $j = n-1$, as a consequence of (1.30) and the fact that

$$\|u\|_D^2 \geqslant \alpha\int_a^b a_n(t)[D^n u(t)]^2\,dt.$$

To show that (1.12) has a solution for $\bar{\Delta}^{(j)}$ sufficiently small, we note that minimizing $B(u,u)$ over $U(\bar{r})$ is equivalent to minimizing $\|u\|_M^2$ over $U(\bar{r})$. But for $\bar{\Delta}^{(j)}$ sufficiently small, H is a Hilbert space under $\|\cdot\|_M$, and $U(\bar{r})$ is a closed and convex subset of H. Thus, (1.12) has a unique solution.

2. Characterization

In cases where M consists only of point evaluations, Λ-splines can be completely characterized as piecewise solutions of $\Lambda u = 0$, satisfying certain generalized continuity conditions. The following definitions and results are extensions of those of [7] to Λ-splines.

Definition 3. We say that $M = \{\mu_i\}_{i=1}^m$ generates an Hermite–Birkhoff (HB) interpolation problem if to each $\mu_i \in M$ there corresponds a pair (x_i, j_i) such that $\mu_i f \equiv D^{j_i} f(x_i)$, where $a \leqslant x_i \leqslant b$ and $0 \leqslant j_i \leqslant n-1$. If for each $\mu_i \in M$,

$$\mu_i f \equiv \sum_{j=0}^{n-1} \alpha_{ij} D^j f(x_i),$$

where the α_{ij} are real and the vectors $\alpha_i \equiv (\alpha_{i,0}, \ldots, \alpha_{i,n-1})$ defining functionals associated with the same point are linearly independent, then we say that M generates an Extended Hermite–Birkhoff (EHB) interpolation problem.

We remark that an (HB) problem is a special case of an (EHB) problem. Moreover, by definition, the linear functionals defining an (EHB) problem are linearly independent, and, as a consequence of (1.9), continuous over H.

Let M generate a Λ-poised (EHB) interpolation problem. We wish to state and prove a theorem characterizing $Sp(\Lambda, M)$. The point $x \in [a, b]$ is said to be a *knot* of $s \in Sp(\Lambda, M)$ if

$$\mu f \equiv \sum_{i=1}^{n-1} \alpha_i D^i f(x)$$

is in M for some choice of $\{\alpha_i\}_{i=1}^{n-1}$.

We first show that s satisfies $\Lambda s(x) \equiv 0$ in the intervals between the knots $a \leqslant x_1 < x_2 < \cdots < x_k \leqslant b$ of s. Let $g \in C_c^\infty(x_i, x_{i+1})$. Since M generates an (EHB) problem, then $g \in U(\bar{0})$. Hence, from (1.20), after integration by parts

$$0 = B(s, g) = \int_{x_i}^{x_{i+1}} s(x) \cdot \Lambda g(x)\, dx. \tag{2.1}$$

Since $a_n(x) > 0$ for $x \in (a, b)$ it then follows from well-known arguments [6] that $\Lambda s = 0$ on (x_i, x_{i+1}).

Let $x \in (a, b)$, $s \in Sp(\Lambda, M)$ and $g \in U(\bar{0}) \cap C_c^\infty(x - \epsilon, x + \epsilon)$. Then from (1.20), we have that

$$0 = \int_{x-\epsilon}^{x} \sum_{j=0}^{n} a_j(x)\, D^j s(x)\, D^j g(x)\, dx + \int_{x}^{x+\epsilon} \sum_{j=0}^{n} a_j(x)\, D^j s(x)\, D^j g(x)\, dx. \tag{2.2}$$

Defining

$$O_i s \equiv \sum_{j=0}^{n-i-1} (-1)^{j+1} D^j[a_{j+i+1} D^{j+i+1} s], \tag{2.3}$$

then, upon integration by parts we have that

$$0 = \sum_{i=0}^{n-1} D^i g(x) [O_i s]_x, \tag{2.4}$$

where $[f]_x \equiv f(x+) - f(x-)$. The same relation holds at $x = a$ or $x = b$, with $[f]_a \equiv f(a+)$ and $[f]_b \equiv -f(b-)$ provided the appropriate limits exist.

Let $\alpha = (\alpha_{ij})_{0,0}^{l-1,n-1}$, $l \leqslant n$ be of rank l and let $\tilde{\alpha}$ be an $n \times n$ nonsingular matrix obtained by augmenting α. Let $\eta \equiv (\eta_{ij})$ be the inverse of the adjoint of $\tilde{\alpha}$. Then we have the following easy lemma [cf. 7].

LEMMA. *Let*

$$M_i = \sum_{j=0}^{n-1} \tilde{\alpha}_{ij} \left(\frac{d}{dx}\right)^j \qquad \textit{and} \quad R_i = \sum_{j=0}^{n-1} \eta_{ij} O_j\,, \qquad 0 \leqslant i \leqslant n-1.$$

Then

$$\sum_{i=0}^{n-1} D^i g(x)[O_i s]_x = \sum_{i=0}^{n-1} M_i g(x)[R_i s]_x\,, \tag{2.5}$$

for all $g \in H$ *and* $s \in Sp(\Lambda, M)$.

Suppose $x \in [a, b]$ is a knot of s. Then there are $l(x)$, $1 \leqslant l(x) \leqslant n$, linear functionals in M of the form

$$M_i^{(x)} s = \sum_{j=0}^{n-1} \alpha_{ij}(x)\, D^j s(x), \qquad 0 \leqslant i \leqslant l(x) - 1, \tag{2.6}$$

where the $\{\alpha(x) = (\alpha_{i,0}(x),..., \alpha_{i,n-1}(x)\}_0^{l(x)-1}$ are linearly independent. We denote as $[R_i^{(x)} s]_x$ the quantity satisfying (2.5) with the M_i as in (2.6).

We now state and prove a characterization theorem for Λ-splines.

THEOREM 4. *Let* $M = \{\mu_i\}_{i=1}^m$ *generate a* Λ*-poised* (EHB) *interpolation problem. Let* $s \in Sp(\Lambda, M)$ *interpolate* $(r_1, ..., r_m)^T$ *with respect to* M. *Then*

$$\begin{array}{lll} \text{(i)}\ \Lambda s(x) = 0, & \text{if } x \text{ is not a knot,} & \\ \text{(ii)}\ \mu_i s = r_i\,, & 1 \leqslant i \leqslant m, & \\ \text{(iii)}\ [R_i^{(x)} s]_x = 0, & l(x) \leqslant i \leqslant n-1, \text{ if } x \text{ is a knot,} & (2.7) \\ \text{(iv)}\ [O_i s]_a = 0, & 0 \leqslant i \leqslant n-1 \text{ if } a \text{ is not a knot,} & \\ \text{(v)}\ [O_i s]_b = 0, & 0 \leqslant i \leqslant n-1 \text{ if } b \text{ is not a knot,} & \end{array}$$

provided the limits in (iv) *and* (v) *exist and, if* $x = a$ *or* b, *the limits in* (iii). *Conversely, if* $s \in H$ *satisfies* (2.7), *then* s *is a* Λ*-spline interpolating* (EHB) *data* $(r_1, ..., r_m)^T$, *provided* (1.20) *implies* (1.12) *has a solution.*

Proof. (2.7 (i), (ii)) have already been proven. (2.7 (iv), (v)) follow directly from (2.4). To prove (2.7 (iii)), assume $x \in (a, b)$, and fix j, $l(x) \leqslant j \leqslant n-1$. Choose $\epsilon > 0$ such that x is the only possible knot in $(x - \epsilon, x + \epsilon)$. There exists a function $g \in C_c^\infty(x - \epsilon, x + \epsilon)$ satisfying

$(g(x),..., g^{n-1}(x))^T = (\tilde{\alpha})^{-1} I_j$, where I_j is the j-th column of the $n \times n$ unit matrix. Thus, by construction, $M_j^{(i)} g(x) = \delta_{ij}$, $0 \leqslant i \leqslant n-1$, and $g \in U(\bar{0})$. The result of (2.7) then follows from (2.4) and (2.5). The cases when a and/or b are knots are handled similarly.

The converse follows from the fact that (2.7) implies that the orthogonality relation of (1.20) is satisfied.

Since $s \in C^{n-1}[a, b]$, we have for, $x \in [a, b]$

$$([O_{n-1}s]_x ,..., [O_0 s]_x)^T = \zeta([D^n s]_x ,..., [D^{2n-1}s]_x)^T,$$

where ζ is a lower triangular matrix with $\pm a_n(x)$ on the diagonal. Since $([R_0^{(x)}s]_x ,..., [R_{n-1}^{(x)}s]_x)^T = \eta([O_0 s]_x ,..., [O_{n-1}s]_x]^T$, then the equations (2.7 (iii)) represent $n - l(x)$ linearly independent relations among the $\{[s^j]_x\}_n^{2n-1}$ at the point x (except at $x = a$ or $x = b$).

COROLLARY 3. *If s is a Λ-spline corresponding to an* (HB) *interpolation problem, and the l-th derivative* $(0 \leqslant l \leqslant n-1)$ *evaluated at the knot x is not involved in the* (HB) *data, then* $[O_l s]_x = 0$, *where the limit is assumed to exist if $x = a$ or b.*

COROLLARY 4. *Let s be a Λ-spline corresponding to an* (HB) *interpolation problem, and suppose ν denotes the order of the highest derivative specified at a knot $x \in (a, b)$. Then $[D^j s]_x = 0$ for $0 \leqslant j \leqslant 2n - 2 - \nu$.*

In the special case $\Lambda = (-1)^n D^{2n}$, we have the following.

COROLLARY 5. *Let s be a Λ-spline with $\Lambda = (-1)^n D^{2n}$, interpolating* (HB) *data; and suppose the l-th derivative* $0 \leqslant l \leqslant n-1)$ *is not specified at a knot x. Then* $[D^{2n-l-1}s]_x = 0$.

3. ERROR ESTIMATES

In this section we shall obtain error estimates for the approximation of smooth functions by Λ-splines. We first make some definitions.

DEFINITION 4. $\tilde{f} \in Sp(\Lambda, M)$ is called an $Sp(\Lambda, M)$-interpolate of $f \in H$ if $\mu\tilde{f} = \mu f$, for all $\mu \in M$.

DEFINITION 5. Let M generate an (EHB) interpolation problem, and $\{x_j\}$ be the corresponding knots. The subset Δ consisting of all $x \in \{x_j\}$ such that there exists $\mu \in M$ satisfying $\mu f = f(x)$ is called the *partition* of $[a, b]$ induced by M. If Δ is not empty, define $\bar{\Delta}$ as the maximum length of the subintervals into which $[a, b]$ is decomposed by the points of Δ, and $\underline{\Delta}$ as the minimum such length. If $x \in \Delta$, let $i(x)$ be defined as the maximum positive integer such that there exists $\mu_k \in M$ for which $\mu_k f = D^k f(x)$ for *each* $0 \leqslant k \leqslant i(x) - 1$, and define $\gamma(\Delta)$ by

$$\gamma(\Delta) = \sum_{x \in \Delta} i(x).$$

THEOREM 5. *Let $f \in H$, let M generate a Λ-poised* (EHB) *interpolation problem, and assume $\gamma(\Delta) \geqslant n$. If s is an $Sp(\Lambda, M)$-interpolate of f, then $B(f-s, f-s)$ is nonnegative for $\bar{\Delta}$ sufficiently small and there exist positive constants c_j, $0 \leqslant j \leqslant n-1$, independent of f and Δ, such that*

$$\| D^j(f-s)\|_{L^\infty[a,b]} \leqslant c_j \left[w_1\left(\frac{1}{a_n}\right), n\bar{\Delta}\right]^{1/2} \cdot (\bar{\Delta})^{n-j-1} [B(f-s, f-s)]^{1/2}. \tag{3.1}$$

In addition, if the hypotheses of Theorem 3 *are satisfied for the j_0 of* (1.7) *then*

$$\| D^j(f-s)\|_{L^\infty[a,b]} \leqslant \tilde{c}_j \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} (\bar{\Delta})^{n-j-1} \|f\|_D, \qquad 0 \leqslant j \leqslant n-1,$$

for $\bar{\Delta}^{(j_0)}$ sufficiently small, where

$$w_1(g, \delta) \equiv \sup_{\substack{x, x+t \in [a,b] \\ 0 \leqslant |t| \leqslant \delta}} \left| \int_x^{x+t} | g(t)| \, dt \right|.$$

Proof. Let $\Delta = \{\xi_0 < \xi_1 < \cdots < \xi_N\}$, and s be any $Sp(\Lambda, M)$ interpolate of f. (s is not necessarily unique.) Since $f - s \in C^{n-1}[a, b]$, and $\gamma(\Delta) \geqslant n$, we can apply a generalized Rolle's theorem, i.e., setting $\xi_i^{(0)} = \xi_i$, $0 \leqslant i \leqslant N$, there exist points $\Delta^{(j)} = \{\xi_l^{(j)}\}_0^{N_j}$ in $[a, b]$ such that

$$D^j f(\xi_l^{(j)}) - D^j s(\xi_l^{(j)}) = 0, \qquad 0 \leqslant l \leqslant N_j, \qquad 0 \leqslant j \leqslant n-1, \tag{3.2}$$

where $N = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_{n-1} \geqslant 0$, where the points of $\Delta^{(j)}$ satisfy

$$a \leqslant \xi_0^{(j)} < \xi_1^{(j)} < \cdots < \xi_{N_j}^{(j)} \leqslant b$$

and

$$\xi_l^{(j)} \leqslant \xi_l^{(j+1)} < \xi_{l+1}^{(j)} \qquad \text{for all} \quad 0 \leqslant l \leqslant N_j \qquad \text{and} \quad 0 \leqslant j \leqslant n-2.$$

It follows immediately that $| \xi_{l+1}^{(j)} - \xi_l^{(j)} | \leqslant (j+1)\bar{\Delta}$, $| a - \xi_0^{(j)} | \leqslant (j+1)\bar{\Delta}$, and $| b - \xi_{N_j}^{(j)} | \leqslant (j+1)\bar{\Delta}$ for any $0 \leqslant j \leqslant n-1$. For each such j, let $x_j \in [a, b]$ be such that

$$| D^j(f(x_j) - s(x_j))| = \| D^j(f-s)\|_{L^\infty}, \qquad 0 \leqslant j \leqslant n-1. \tag{3.3}$$

Since $\overline{\Delta^{(j)}} \equiv \max_i | \xi_{i+1}^{(j)} - \xi_i^{(j)} | \leqslant (j+1)\bar{\Delta}$, there is a point $\xi_k^{(j)}$ such that $| x_j - \xi_k^{(j)} | \leqslant (j+1)\bar{\Delta}$. From (3.2) and (3.3), therefore,

$$\| D^j(f-s)\|_{L^\infty} = \left| \int_{\xi_k^{(j)}}^{x_j} D^{j+1}(f(t)-s(t))\, dt \right|, \qquad 0 \leqslant j \leqslant n-1. \tag{3.4}$$

For $0 \leqslant j \leqslant n-2$, then

$$\| D^j(f-s)\|_{L^\infty} \leqslant (j+1)\,\bar{\Delta}\,\| D^{j+1}(f-s)\|_{L^\infty},$$

and we have that

$$\| D^j(f-s)\|_{L^\infty} \leqslant \frac{(n-1)!}{(j)!}\,(\bar{\Delta})^{n-j-1}\,\| D^{n-1}(f-s)\|_{L^\infty}, \qquad 0 \leqslant j \leqslant n-2. \tag{3.5}$$

From (3.4) and the Schwarz inequality,

$$\| D^{n-1}(f-s)\|_{L^\infty}^2 = \left| \int_{\xi_k^{(n-1)}}^{x_{n-1}} D^n(f(t)-s(t))\, dt \right|^2$$

$$\leqslant \left[\int_{\xi_k^{(n-1)}}^{x_{n-1}} \left| \frac{1}{\sqrt{a_n(t)}} \cdot \sqrt{a_n(t)}\, D^n[f(t)-s(t)] \right| dt \right]^2$$

$$\leqslant \int_{\xi_k^{(n-1)}}^{x_{n-1}} \frac{1}{a_n(t)}\, dt \cdot \int_{\xi_k^{(n-1)}}^{x_{n-1}} a_n(t)[D^n[f(t)-s(t)]]^2\, dt.$$

Therefore,

$$\| D^{n-1}(f-s)\|_{L^\infty}^2 \leqslant w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right) \cdot \int_a^b a_n(t)[D^n[f(t)-s(t)]]^2\, dt. \tag{3.6}$$

But, from (1.10) and (1.11), there exists a positive constant K such that

$$\int_a^b a_n(t)[D^n(f(t)) - s(t))]^2\, dt$$

$$\leqslant K \| f-s \|_D^2 = K[B(f-s, f-s) + C \| f-s \|_{j_0,2}^2]. \tag{3.7}$$

Since

$$\| D^k(f-s)\|_{L^2}^2 \leqslant (b-a)\| D^k(f-s)\|_{L^\infty}^2 \qquad 0 \leqslant k \leqslant n-1,$$

we have from (3.5)–(3.7) that

$$\int_a^b a_n(t)[D^n[f(t) - s(t)]]^2\, dt$$
$$\leqslant KB(f-s, f-s) + KC(b-a) \sum_{k=0}^{j_0} \left[\frac{(n-1)!}{k!} (\bar{\Delta})^{n-k-1}\right]^2$$
$$\times\, w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right) \cdot \int_a^b a_n(t)[D^n[f-s]]^2\, dt$$

For $\bar{\Delta}$ sufficiently small, the coefficient of the second term on the right-hand side can be made less than 1/2 for *any* $j_0 \leqslant n-1$ (since $1/a_n \in L^1$). Thus

$$\int_a^b a_n(t)[D^n(f(t) - s(t))]^2\, dt \leqslant 2KB(f-s, f-s) \tag{3.8}$$

for all $\bar{\Delta}$ sufficiently small. Equations (3.5), (3.6) and (3.8) then give

$$\| D^j(f-s)\|_{L^\infty}$$
$$\leqslant \sqrt{2K}\, \frac{(n-1)!}{j!} (\bar{\Delta})^{n-j-1} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} [B(f-s, f-s)]^{1/2}$$
$$0 \leqslant j \leqslant n-1 \tag{3.9}$$

for all $\bar{\Delta}$ sufficiently small.

To prove the second inequality in (3.1) we recall from the proof of Theorem 3 that, if $\bar{\Delta}^{(j_0)}$ is sufficiently small, then the norm

$$\|f\|_M^2 = B(f,f) + \sum_{\mu \in M} [\mu(f)]^2, \qquad \text{for all } \ f \in H, \tag{3.10}$$

is a norm on H equivalent to the norm, $\|\cdot\|_D$. Thus, since,

$$B(f-s, f-s) = \|f-s\|_M^2 = \|f\|_M^2 - \|s\|_M^2 \leqslant \|f\|_M^2 \leqslant K\|f\|_D^2 \tag{3.11}$$

the second inequality follows from the first. This completes the proof of the theorem. We now obtain L_2-norm error estimates.

THEOREM 6. *Let $f \in H$, and let M generate a Λ-poised* (EHB) *interpolation problem. We assume that $\gamma(\Delta) \geqslant n$. If $s \in Sp(\Lambda, M)$ interpolates f, then, for $\bar{\Delta}$ sufficiently small, there exist positive constants $c_j^{(1)}$, $0 \leqslant j \leqslant n-1$, independent of f and Δ, such that*

$$\| D^j(f-s)\|_{L^2} \leqslant c_j^{(1)} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} (\bar{\Delta})^{n-j-1/2}[B(f-s, f-s)]^{1/2}. \tag{3.12}$$

In addition, if the hypotheses of Theorem 3 are satisfied for the j_0 of (1.7) then

$$\| D^j(f-s)\|_{L^2} \leqslant \tilde{c}_j^{(1)} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} (\bar{\Delta})^{n-j-1/2} \|f\|_D, \qquad 0 \leqslant j \leqslant n-1,$$

for $\bar{\Delta}^{(j_0)}$ sufficiently small and

$$\int_a^b a_n(x)[D^n(f-s)(x)]^2\,dx \leqslant c_n^{(1)} B(f-s, f-s) \leqslant \tilde{c}_n^{(1)} \|f\|_D^2. \qquad (3.12')$$

Proof. For any $0 \leqslant j \leqslant n-1$, we have from (3.2) that $D^j(f-s)$ vanishes at $\xi_l^{(j)}$ for $0 \leqslant l \leqslant N_j$. Applying the Rayleigh–Ritz inequality we have that

$$\int_{\xi_l^{(j)}}^{\xi_{l+1}^{(j)}} [D^j(f-s)(t)]^2\,dt \leqslant \left[\frac{(j+1)\,\bar{\Delta}}{\pi}\right]^2 \int_{\xi_l^{(j)}}^{\xi_{l+1}^{(j)}} [D^{j+1}(f-s)(t)]^2\,dt,$$
$$0 \leqslant l \leqslant N_j, \qquad 0 \leqslant j \leqslant n-2, \quad (3.13)$$

since $\xi_{l+1}^{(j)} - \xi_l^{(j)} \leqslant (j+l)\bar{\Delta}$. Summing both sides with respect to l, we have that

$$\int_{\xi_0^{(j)}}^{\xi_{N_j}^{(j)}} [D^j(f-s)(t)]^2\,dt \leqslant \left[\frac{(j+1)\,\bar{\Delta}}{\pi}\right]^2 \| D^{j+1}(f-s)\|_{L^2}^2, \qquad 0 \leqslant j \leqslant n-2. \quad (3.14)$$

For $j = n-1$, we cannot use the Rayleigh–Ritz inequality, since $D^n(f-s)$ is not necessarily in L^2. However,

$$\begin{aligned}
&\| D^{n-1}(f-s)\|_{L^2[a,b]}^2 \\
&\quad = \| D^{n-1}(f-s)\|_{L^2[a,\xi_0^{(n-1)}]}^2 \\
&\qquad + \sum_{k=0}^{N_{n-1}-1} \| D^{n-1}(f-s)\|_{L^2[\xi_k^{(n-1)},\, \xi_{k+1}^{(n-1)}]}^2 + \| D^{n-1}(f-s)\|_{L^2[\xi_{N_{n-1}}^{(n-1)},\, b]}^2.
\end{aligned} \quad (3.15)$$

For each x in the subinterval $[\xi_k^{(n-1)}, \xi_{k+1}^{(n-1)}]$,

$$\begin{aligned}
D^{n-1}(f-s)(x) &= \int_{\xi_k^{(n-1)}}^x D^n(f-s)(t)\,dt \\
&= \int_{\xi_k^{(n-1)}}^x \frac{1}{\sqrt{a_n(t)}} \sqrt{a_n(t)}\, D^n(f-s)(t)\,dt.
\end{aligned}$$

Thus,

$$[D^{n-1}(f-s)(x)]^2 \leqslant w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right) \int_{\xi_k^{(n-1)}}^{\xi_{k+1}^{(n-1)}} a_n[D^n(f-s)(t)]^2\, dt$$

and, integrating, we get that

$$\| D^{n-1}(f-s)\|^2_{L^2[\xi_k^{(n-1)},\, \xi_{k+1}^{(n-1)}]}$$

$$\leqslant w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)(n\bar{\Delta}) \cdot \int_{\xi_k^{(n-1)}}^{\xi_{k+1}^{(n-1)}} a_n(t)[D^n(f-s)(t)]^2\, dt \tag{3.16}$$

for each $0 \leqslant k \leqslant N_{n-1} - 1$. However, for each $0 \leqslant j \leqslant n-1$,

$$\int_a^{\xi_0^{(j)}} [D^j(f-s)(t)]^2\, dt \leqslant | \xi_0^{(j)} - a \| D^j(f-s)\|^2_{L^\infty},$$

and using the fact that $| \xi_0^{(j)} - a | \leqslant (j+1)\bar{\Delta}$, as well as (3.5) and (3.6),

$$\int_a^{\xi_0^{(j)}} [D^j(f-s)(t)]^2\, dt$$

$$\leqslant c_j''(\bar{\Delta})^{2n-2j-1} w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right) \int_a^b a_n(t)[D^n[f-s](t)]^2\, dt, \tag{3.17}$$

with the similar inequality

$$\int_{\xi_{N_j}^{(j)}}^b [D^j(f-s)(t)]^2\, dt$$

$$\leqslant c_j''(\bar{\Delta})^{2n-2j-1} w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right) \int_a^b a_n(t)[D^n[f-s](t)]^2\, dt. \tag{3.18}$$

From (3.15)–(3.18), we then have that

$$\| D^{n-1}(f-s)\|_{L^2}$$

$$\leqslant C_{n-1}^{(1)}(\bar{\Delta})^{1/2}\left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2}\left\{\int_a^b a_n(t)[D^n[(f-s)(t)]^2\, dt]\right\}^{1/2}, \tag{3.19}$$

and using (3.8) and (3.11) the result of (3.12) follows for $j = n-1$. The inequality (3.12) for $0 \leqslant j \leqslant n-2$ follows by induction using (3.14), (3.17) and (3.18). The result of (3.12′) follows directly from (3.8) and (3.11).

Suppose M generates a Λ-poised (EHB) interpolation problem. Let f be such that $\Lambda f \in L^2[a, b]$ (and also $f \in H$), and let s be an $Sp(\Lambda. M)$-interpolate of f. Then, from Theorem 4, we have, upon integration by parts, that

$$B(f - s, f - s) = \int_a^b (f - s)(x) \cdot \Lambda f(x)\, dx + \left[\sum_{i=0}^{n-1} D^i(f - s)\, O_i(f - s)\right]_{a+}^{b-} \tag{3.20}$$

whenever the limits $[O_i(f - s)]_{a+}^{b-}$ exist, $0 \leqslant i \leqslant n - 1$, and

$$\begin{aligned} M \supset \{\mu_j : \mu_j f &\equiv D^j f(a), \quad 0 \leqslant j \leqslant n - 1\} \\ \cup \{\mu_j : \mu_j f &\equiv D^j f(b), \quad 0 \leqslant j \leqslant n - 1\}. \end{aligned}$$

Then (3.20) becomes

$$B(f - s, f - s) = \int_a^b (f - s)(x) \cdot \Lambda f(x)\, dx, \tag{3.21}$$

which is known in the literature as the second integral relation. Higher order error estimates for spline interpolation in general require the assumption of the second integral relation.

THEOREM 7. *Let $f \in H$, $\Lambda f \in L^2$, and let M generate a Λ-poised* (EHB) *interpolation problem. We assume that $\gamma(\Delta) \geqslant n$, and that the second integral relation of* (3.21) *is satisfied. If $s \in Sp(\Lambda, M)$ interpolates f, then for $\bar{\Delta}$ sufficiently small, there exist positive constants $c_j^{(2)}$, independent of f and Δ, such that*

$$\| D^j(f - s)\|_{L^\infty} \leqslant c_j^{(2)}(\bar{\Delta})^{2n-j-3/2} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right] \| \Lambda f\|_{L^2}, \qquad 0 \leqslant j \leqslant n - 1. \tag{3.22}$$

Proof. From (3.21) and the Schwarz inequality,

$$B(f - s, f - s) \leqslant \| f - s\|_{L^2} \| \Lambda f\|_{L^2}.$$

But, from (3.12),

$$B(f - s, f - s) \leqslant c_0^{(1)} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} (\bar{\Delta})^{n-1/2}[B(f - s, f - s)]^{1/2} \| \Lambda f\|_{L^2}.$$

Thus,

$$[B(f - s, f - s)]^{1/2} \leqslant c_0^{(1)} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} (\bar{\Delta})^{n-1/2} \| \Lambda f\|_{L^2}. \tag{3.23}$$

Using this in (3.1) then gives the result of (3.22).

THEOREM 8. *Let f, M, s, and Δ satisfy the hypotheses of Theorem 7.*

Then, for $\bar{\Delta}$ sufficiently small, there exist positive constants $c_j^{(3)}$, $0 \leqslant j \leqslant n-1$ independent of f and Δ, such that

$$\| D^j(f-s)\|_{L^2} \leqslant c_j^{(3)}(\bar{\Delta})^{2n-j-1} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right] \|\Lambda f\|_{L^2}, \qquad 0 \leqslant j \leqslant n-1,$$

and if the hypotheses of Theorem 3 *are satisfied for the j_0 of* (1.7) *then*

$$\|f-s\|_M \leqslant c_n^{(3)}(\bar{\Delta})^{n-1/2} \left[w_1\left(\frac{1}{a_n}, n\bar{\Delta}\right)\right]^{1/2} \|\Lambda f\|_{L^2} \tag{3.24}$$

for $\bar{\Delta}^{(j_0)}$ sufficiently small where $\|\cdot\|_M$ is as defined in Theorem 3.

Proof. Follows Directly from (3.23), (3.12) and (1.27).

Remarks. (1) We wish to discuss briefly how the present paper extends the concept of Lg-spline as contained in [7] and [8]. There, Λ was the differential operator L^*L, and $a_n(x) \geqslant \alpha > 0$ for all $x \in [a, b]$.

First, we point out that for any Λ, satisfying (1.1) and (1.2), and any bilinear form B, satisfying (1.6) and the property that Λ is the associated Euler operator, the entire theory that we have developed for the special choice of B in (1.3) carries over, with obvious modifications in the definition of the operators O_i in (2.3). In particular, if Λ is of the form L^*L then the bilinear form

$$B'(u, u) = \int_a^b [Lu(x)]^2\, dx$$

clearly has Λ as its Euler operator and, in addition, may readily be shown to satisfy (1.6) with B' in place of B. Here, it is not necessary, even, that L be nonsingular. We require only that L^*L satisfy (1.1) and (1.2). Now, since the form B' is obviously nonnegative, Theorem 1 implies the existence of the spline in this case. If, in addition, L is nonsingular, the condition for uniqueness of the spline in Theorem 1 reduces to the familar condition

$$N_L \cap U(\bar{0}) = (0),$$

where N_L is the null space of L. This follows easily from the readily verifiable fact that (1.18) is equivalent, in this case, to

$$\begin{aligned} &(L^*L + CI)\,\varphi = \lambda\varphi \\ &\qquad D^jL\varphi(a) = D^jL\varphi(b) = 0, \qquad 0 \leqslant j \leqslant n-1, \end{aligned} \tag{3.25}$$

which implies that C is an eigenvalue of (3.25) with corresponding eigenspace precisely N_L.

It is, of course, essential that forms more general than (1.3) be considered. In particular, if $\Lambda = -D^2 + I$, then $\Lambda = L^*L$, with $L = D + I$, and

$$B'(u, u) = \int_a^b [Lu(x)]^2 \, dx$$
$$= \int_a^b [Du(x)]^2 \, dx + 2 \int_a^b u(x) \, Du(x) \, dx + \int_a^b [u(x)]^2 \, dx.$$

On the other hand, the form B, defined by (1.3), yields in this case,

$$B(u, u) = \int_a^b [Du(x)]^2 \, dx + \int_a^b [u(x)]^2 \, dx$$

which is also nonnegative. The functions minimizing the functionals B and B' are not, in general, the same of course.

The γ-elliptic splines of Schultz [11] are readily attainable by our methods. There it was assumed that Λ was nonsingular and coercive over the class H_0 of functions in H whose derivatives through order $n - 1$ at a and b vanished, i.e., B defined a norm equivalent to the Sobolev norm on H_0; splines satisfying Hermite data were then defined, locally annihilated by Λ, where derivatives through order $n - 1$ were specified at a and b. Our own arguments show that such splines can be obtained, since in this case B is nonnegative, over H_0; we simply minimize B over H_0, subject to the Hermite constraints corresponding to interior mesh points, with data decreased by the values $D^j\varphi(x_i)$, where φ is the unique member of the null space of Λ satisfying the $2n$ endpoint conditions. $S = u + \varphi$, where u solves the minimization problem, is the γ-elliptic spline.

(2) We shall now include an example to illuminate the previous theory. For simplicity, we choose points $0 < x_1 < \cdots < x_m < 1$ and $0 < \sigma < 1$. We consider the minimization problem

$$\min \left\{ \int_0^1 x^\sigma [D^n u(x)]^2 \, dx : u(x_j) = r_j, \qquad 1 \leqslant j \leqslant m \right\} \tag{3.26}$$

in the space $H = \{u : \int_0^1 x^\sigma [D^n u(x)]^2 dx = B(u, u) < \infty\}$. $B(u, u)$ is evidently nonnegative and by Theorem 1 a solution s exists which is unique if $m \geqslant n$. Furthermore, $s \in C^{2n-2}[0, 1] \cap C^{2n}((0, x) \cup (x_1, x_2) \cdots \cup (x_m, 1))$ and satisfies, on each of the subintervals $(0, x_1)$, (x_1, x_2),... $(x_m, 1)$, the differential equation

$$(-1)^n D^n (x^\sigma D^n s(x)) = 0. \tag{3.27}$$

This implies that s is of the form

$$s(x) = \alpha_0 x^{2n-1-\sigma} + \alpha_1 x^{2n-2-\sigma} + \cdots + \alpha_{n-1} x^{n-\sigma} + P_{n-1}(x), \tag{3.28}$$

where P_{n-1} is a polynomial of degree $n - 1$, on each of the subintervals. Moreover, on $(0, x_1)$ and $(x_m, 1)$, s actually reduces to a polynomial of degree $n - 1$, i.e., the coefficients $\alpha_0, ..., \alpha_{n-1}$ are zero for these intervals. This is not the case, however, if certain derivatives of order j, $0 \leqslant j \leqslant n - 1$, are specified at 0 and 1. In this latter case, one can say that, if the j-th derivative is not specified at 0(1), then $D^{n-j-1}(x^\sigma D^n s)(x)$ approaches zero as x tends to 0(1). In particular, this implies that $\alpha_j = 0$ in the representation on $(0, x_1)$.

(3) If the linear equality constraints of Theorem 7 are exclusively of full Hermite type, i.e., derivatives of order through $n - 1$ are specified at each knot, then the exponent of $\bar{\Delta}$ in (3.22) can be improved by one-half unity (see, e.g., Dailey [3]) provided Λf is bounded.

References

1. S. Agmon, "Lectures on Elliptic Boundary Value Problems," Van Nostrand, Princeton, NJ, 1965.
2. P. G. Ciarlet, F. Natterer, and R. S. Varga, Numerical methods of high-order accuracy for singular nonlinear boundary value problems, *Numer. Math.* **15** (1970), 87–99.
3. J. W. Dailey, "Approximation by Spline-Type Functions and Related Problems," dissertation, Case Western Reserve University, Cleveland, Ohio, 1969.
4. M. Golomb, "Splines, n-Widths and Optimal Approximations," M.R.C. Technical Summary Report 784, University of Wisconsin, Madison, WI, 1967.
5. M. Golomb and J. W. Jerome, Linear ordinary equations with boundary conditions on arbitrary point sets, *Trans. Amer. Math. Soc.* **153** (1971), 235–264.
6. I. Halperin, "Theory of Distributions," University of Toronto Press, Toronto, 1952.
7. J. W. Jerome and L. L. Schumaker, On Lg-splines, *J. Approximation Theory* **2** (1969), 29–49.
8. J. W. Jerome and R. S. Varga, Generalizations of spline functions and applications to nonlinear boundary value and eigenvalue problems, *in* "Theory and Applications of Spline Functions" (T. N. E. Greville, Ed.), Academic Press, New York, 1969.
9. T. Lucas, A generalization of L-splines, *Numer. Math.* **15** (1970), 359–370.
10. F. Riesz and B. Sz.-Nagy, "Functional Analysis," Ungar, New York, 1955.
11. M. H. Schultz, Elliptic spline functions and the Rayleigh–Ritz Galerkin method, *Math. Comp.* **24** (1970), 65–80.
12. J. W. Jerome, Singular self-adjoint multipoint boundary-value problems: solutions and approximations, *Int. Ser. Numer. Math.*, to appear.